

Critical sets of the total variance of state detect all SLOCC entanglement classes

Adam Sawicki^{1,2}, Michał Oszmaniec², and Marek Kuś²

¹*School of Mathematics, University of Bristol, University Walk, Bristol BS8 1TW, UK,*

²*Center for Theoretical Physics PAS, Al. Lotników 32/46, 02-668 Warszawa, Poland*

We present a general algorithm for finding all classes of pure multiparticle states equivalent under Stochastic Local Operations and Classical Communication (SLOCC). We parametrize all SLOCC classes by the critical sets of the total variance function. Our method works for arbitrary systems of distinguishable and indistinguishable particles. We also show how to calculate the Morse indices of critical points which have the interpretation of the number of independent non-local perturbations increasing the variance and hence entanglement of a state. We illustrate our method by two examples.

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The problem of classifying pure states up to Stochastic Local Operations assisted by Classical Communication (SLOCC) has been intensely studied during the last decade by many authors [1–4]. Although the SLOCC classes are known for some particular systems, for example in the cases of three or four qubits, the general method allowing similar derivations for arbitrary system of many particles which treats in the unified way distinguishable and indistinguishable particles has been missing. In this letter we provide such an algorithm. It is based on studying the structure of critical points of the total variance of a state with respect to local unitary (LU) operations. Since the total variance of the state can be interpreted as an entanglement measure [5] our approach has a clear physical meaning. Although our results are based on some relatively advanced mathematical tools, the final algorithm we propose reduces the problem to simple computations involving diagonalized reduced density matrices.

The paper is organized as follows. First we give a definition of the total variance of a pure state, $\text{Var}([\psi])$. Next we discuss a connection between the concept of the momentum map and the reduced one-particle density matrices. Following this we show that the total variance of a state is, up to some unimportant additive constant, the norm squared of the momentum map. This observation allows us to identify the critical points of $\text{Var}([\psi])$, i.e. points $[\psi]$ in $\mathbb{P}(\mathcal{H})$ for which $d\text{Var}([\psi]) = 0$, with the critical points of the momentum map. The later turn out to be well understood in Geometric Invariant Theory (GIT) [6, 7]. Using its results we show that for each SLOCC class of pure states the restriction of $\text{Var}([\psi])$ to it attains maximum on exactly one orbit of the local unitary action. This orbit contains the most entangled representatives of a given SLOCC class. Finally, we describe how to calculate the Morse index of $\text{Var}([\psi])$ at critical points and explain that it has a meaningful interpretation as the number of independent non-local perturbations which increase the total variance, $\text{Var}([\psi])$ thus, consequently, also its entanglement. The whole idea is illustrated by familiar examples.

Throughout the paper we consider the system of L identical particles which can be either distinguishable or indistinguishable. For the former the Hilbert space is the tensor product of L copies of \mathbb{C}^N , $\mathcal{H}_d = \mathbb{C}^N \otimes \dots \otimes \mathbb{C}^N$. Local unitary operations are represented by the direct product of L copies of $SU(N)$, $K = SU(N)^{\times L}$, and SLOCC operations by L copies of $SL(N)$, $G = SL(N)^{\times L}$, (see [1–4]). The action of an element (A_1, \dots, A_L) of either K or G on the vector $|\psi\rangle \in \mathcal{H}_d$ is given by

$$|\psi\rangle \mapsto A_1 \otimes \dots \otimes A_L |\psi\rangle. \quad (1)$$

For indistinguishable particles, i.e. for systems of L bosons or L fermions the relevant Hilbert spaces are fully symmetric or fully antisymmetric parts of the full tensor product, i.e. $\mathcal{H}_b = \text{Sym}^L(\mathbb{C}^N)$ and $\mathcal{H}_f = \bigwedge^L(\mathbb{C}^N)$. The local groups corresponding to LU and SLOCC operations are $K = SU(N)$ and $G = SL(N)$ and the action of an element $A \in K$ (or $A \in G$) on $|\psi\rangle \in \mathcal{H}_{b/f}$ is given by (1) with $A_k = A$. Because the global phase factor and the normalization are physically irrelevant it is useful to identify pure states with elements of the complex projective space $\mathbb{P}(\mathcal{H})$. To this end we identify vectors from \mathcal{H} that differ by a nonzero complex scalar factor and denote by $[\psi] \in \mathbb{P}(\mathcal{H})$ the state corresponding to vectors $|\psi\rangle \in \mathcal{H}$ under this identification. Because groups K and G act linearly on \mathcal{H} they also act naturally on $\mathbb{P}(\mathcal{H})$. Two pure states $[\psi], [\phi] \in \mathbb{P}(\mathcal{H})$ are SLOCC equivalent if and only if $|\psi\rangle$ and $|\phi\rangle$ can be connected by the action of G on the level of \mathcal{H} , i.e. $g \cdot |\psi\rangle = e^{i\alpha} |\phi\rangle$, $\alpha \in [0, 2\pi)$, for some $g \in G$. Actions of K and G on \mathcal{H} induce actions of the corresponding Lie algebras \mathfrak{k} and \mathfrak{g} . The action of $\alpha = (\alpha_1, \dots, \alpha_L)$ in either $\mathfrak{su}(N)^{\oplus L}$ or $\mathfrak{sl}(N)^{\oplus L}$ on the vector $|\psi\rangle \in \mathcal{H}$ is given by $\alpha \cdot |\psi\rangle = (\alpha_1 \otimes I_N \otimes \dots \otimes I_N + \dots + I_N \otimes \dots \otimes I_N \otimes \alpha_L) |\psi\rangle$. For indistinguishable particles $\alpha_k = \alpha \in \mathfrak{su}(N)$ (or $\alpha \in \mathfrak{sl}(N)$).

After fixing notation we can define the total variance

of the state $[\psi] \in \mathbb{P}(\mathcal{H})$, first introduced in [5]

$$\text{Var}([\psi]) = \frac{1}{\langle \psi | \psi \rangle} \sum_{i=1}^{\dim K} \left(\langle \psi | X_i^2 | \psi \rangle + \frac{1}{\langle \psi | \psi \rangle} \langle \psi | X_i | \psi \rangle^2 \right), \quad (2)$$

where $\{X_i\}_{i=1}^{\dim K}$ is an orthonormal (with respect to the Hilbert-Schmidt scalar product $(A, B) = \text{Tr}(A^\dagger B)$) basis of \mathfrak{k} represented as matrices acting in \mathcal{H} [18]. The expression (2) is simply the sum of variances of 'local' observables X_i calculated in the state $[\psi]$. It can be checked that $\text{Var}([\psi])$ is K -invariant and attains its minimum precisely on the set of separable states. In fact, it can be used as a measure of entanglement [5].

In order to proceed we briefly describe the concept of the momentum map. This is a map which encodes information about all first integrals of a given classical Hamiltonian system with symmetries. For example, for a Hamiltonian system with the rotational symmetry, i.e. when the symmetry group is $K = SO(3)$, the first integrals are three components of the angular momentum corresponding to the invariance of the system with respect to infinitesimal rotations along axes x, y, z . The infinitesimal rotations generate the Lie algebra $\mathfrak{so}(3)$ of $SO(3)$. There are many possible choices of basis for $\mathfrak{so}(3)$, each corresponds to different choice of rotation axes and each gives three first integrals. The mathematical object which encodes information about all first integrals for all possible choices of generators of $\mathfrak{so}(3)$ is a map $\mu : M \rightarrow \mathfrak{so}(3)^*$, i.e. a map from the phase space M to the space of linear functionals on the Lie algebra $\mathfrak{so}(3)$ of the group $SO(3)$. For every infinitesimal symmetry $\xi \in \mathfrak{so}(3)$ one can find the corresponding first integral by evaluating $\mu_\xi(x) = \mu(x)(\xi)$. This idea can be generalized to arbitrary K and the corresponding map $\mu : M \rightarrow \mathfrak{k}^*$ is called momentum map [8].

It is well known that $\mathbb{P}(\mathcal{H})$ possesses a natural phase space structure [19]. Moreover, for considered composite systems we can treat LU operations as symmetries of the system as they do not change entanglement. Remarkably, the momentum map for both distinguishable and indistinguishable particles has a clear quantum mechanical meaning. It is a map which assigns to each state its reduced one-particle density matrices. For distinguishable particles we have

$$\mu([\psi]) = \left(\rho_1([\psi]) - \frac{1}{N} I_N, \dots, \rho_N([\psi]) - \frac{1}{N} I_N \right), \quad (3)$$

where $\rho_i([\psi])$ is the normalized i -th reduced one-particle density matrix of the vector state $|\psi\rangle$. For indistinguishable particles we have:

$$\mu([\psi]) = \rho_1([\psi]) - \frac{1}{N} I_N. \quad (4)$$

Since $\mu([\psi])$ is an element of \mathfrak{k} we can calculate its

Hilbert-Schmidt norm. It is easy to see that:

$$\|\mu([\psi])\|^2 = \frac{1}{\langle \psi | \psi \rangle^2} \sum_{i=1}^{\dim K} \langle \psi | X_i | \psi \rangle^2 \quad (5)$$

We also notice that $\mathcal{C}_2 = \sum_{i=1}^{\dim K} X_i^2$ is the representation of the second order Casimir operator [9], a generalization to other Lie algebras of the squared total angular momentum relevant for the above mentioned example of rotationally invariant system. The operator \mathcal{C}_2 commutes with each X_i . Since the group K acts on \mathcal{H} irreducibly, it is proportional to the identity operator with the proportionality constant $c = \frac{\langle \psi | \mathcal{C}_2 | \psi \rangle}{\langle \psi | \psi \rangle}$. Making use of this formula and (5) we get the final expression for the total variance of a state:

$$\text{Var}([\psi]) = c - \|\mu([\psi])\|^2. \quad (6)$$

Notice that by (6) the critical points $\text{Var}([\psi])$ are exactly the critical points of $\|\mu([\psi])\|^2$. The latter were intensely studied in GIT context in the 80'. Specifically, the following theorem is the reformulation of the general result [6, 7]

Theorem 1 *Assume that $[\psi]$ is a critical point of $\text{Var} : \mathbb{P}(\mathcal{H}) \rightarrow \mathbb{R}$. Then the restriction of the total variance function Var to the SLOCC class of $[\psi]$, i.e. G -orbit through $[\psi]$, attains its maximum value on a unique K -orbit, which is the orbit $K \cdot [\psi]$ through the state $[\psi]$.*

In general it may happen that a particular G -orbit does not contain any critical point of $\text{Var} : \mathbb{P}(\mathcal{H}) \rightarrow \mathbb{R}$. Nevertheless, as we show in [10], such an orbit always contains in its closure the K -orbit which is critical for $\text{Var}([\psi])$. The mathematical details of this construction are rather subtle and we will not discuss them here. We would like to emphasize, however, that knowing critical K -orbits allows the reconstruction of these G -orbits which do not contain any critical point of Var (see [10]). Under the above assumptions Theorem 1 can be interpreted as the statement saying that the critical sets of the total variance function of a many-particle system parametrize all SLOCC classes of states. Moreover the distinguished local unitary orbit on which Var attains a maximum contains maximally entangled representatives of a given SLOCC class. Although we already know that the total variance cannot increase when we perturb the state from this orbit with SLOCC operations, it is still not clear what happens when the perturbation is non-local. This information is stored in the Morse index of a critical point i.e. the number of negative eigenvalues of the Hessian of $\|\mu([\psi])\|^2$ calculated at $[\psi]$. We will investigate this idea on two examples. First, however we need some effective way for finding critical points of $\|\mu([\psi])\|^2$ and calculating the Morse index. In [10] we prove the following theorem

Theorem 2 1. The state $|\psi\rangle$ is a critical point of $\text{Var} : \mathbb{P}(\mathcal{H}) \rightarrow \mathbb{R}$ if and only if the corresponding vector $|\psi\rangle$ is an eigenvector of $\mu([\psi])$, i.e. for some $\lambda \in \mathbb{R}$ we have

$$\mu([\psi])|\psi\rangle = \lambda|\psi\rangle. \quad (7)$$

2. The Morse index of $\|\mu([\psi])\|^2$ at a critical point $[\psi]$ is the the Morse index of the function $f : \mathcal{H} \rightarrow \mathbb{R}$,

$$f(|\phi\rangle) = \frac{\langle \phi | \mu([\psi]) | \phi \rangle}{\langle \phi | \phi \rangle}, \quad (8)$$

calculated at $|\phi\rangle = |\psi\rangle$ and restricted to the orthogonal complement (with respect to the usual inner product on \mathcal{H}) of the subspace $\mathfrak{g} \cdot |\psi\rangle$

In Equations (7) and (8) $\mu([\psi])$ is understood as an operator acting on \mathcal{H} . Expression $\mathfrak{g} \cdot |\psi\rangle$ denotes the image of the vector $|\psi\rangle$ under the action of the whole \mathfrak{g} represented on \mathcal{H} , where $\mathfrak{g} = \mathfrak{sl}(N)^{\oplus L}$ for distinguishable particles and $\mathfrak{g} = \mathfrak{sl}(N)$ for indistinguishable particles.

Using (7) we can divide the problem of finding critical points of $\text{Var}([\psi])$ into two conceptually different parts. First, we notice that condition (7) is automatically satisfied if $\mu([\psi]) = 0$. This corresponds to states with maximally mixed reduced one-particle density matrices. Since the set $\mu^{-1}(0)$ is K -invariant, each K -orbit contained in $\mu^{-1}(0)$ represents a different SLOCC class of states. By splitting $\mu^{-1}(0)$ into disjoint orbits of K and picking exactly one state from each orbit we get a parametrization of all SLOCC classes of this type. These classes contain almost all states [10]. For effective calculations it is desirable to have some canonical form of the state. In N -particle case such a form was given in [11]. The remaining critical points of $\text{Var}(\psi)$ stem from (7) when $\mu([\psi]) \neq 0$. The following two remarks are crucial in this case (see [10] for a detailed discussion)

1. For any $|\psi\rangle$ there exists LU operator $U = U_1 \otimes \dots \otimes U_L$ such that $\mu([\psi'])$, where $|\psi'\rangle = U|\psi\rangle$, is given by (3) with diagonal reduced density matrices $\rho_i([\psi'])$ whose eigenvalues are arranged in non-increasing order; the matrices $U_i \in SU(N)$ diagonalize $\rho_i([\psi])$. Notice that since critical sets of $\text{Var}([\psi])$ are K -invariant it implies that we can consider equation (7) assuming that ψ posses the above described property of ψ' . For bosons and fermions the relevant operators U also exist and are given by $U = U_1 \otimes \dots \otimes U_1$, where $U_1 \in SU(N)$ diagonalizes $\rho_1([\psi])$ from (4) and makes its spectrum non-increasingly ordered.

2. Let $P_i(\psi)$ be a point in \mathbb{R}^N whose coordinates are given by non-increasingly ordered spectrum of $\rho_i([\psi]) - \frac{1}{N}I_N$. The set $\Psi(\mathbb{P}(\mathcal{H})) = \{(P_1(\psi), \dots, P_L(\psi)) : [\psi] \in \mathbb{P}(\mathcal{H})\}$ is a convex polytope in \mathbb{R}^{LN} , the so-called Kirwan or momentum polytope. This fact is a simple corollary [12] from the general theorem known as the convexity property of the momentum map [13]. Moreover, in the

variety of settings for distinguishable and indistinguishable particles the inequalities defining this polytope are explicitly known [14, 15].

Using the above remarks the search for remaining critical points of $\text{Var}([\psi])$ can be reduced to the following procedure. For each point $P = (P_1, \dots, P_L)$ in the polytope $\Psi(\mathbb{P}(\mathcal{H})) \setminus \{0\}$ we construct corresponding operator

$$\alpha_P = \alpha_{P_1} \otimes I_N \otimes \dots \otimes I_N + \dots + I_N \otimes \dots \otimes I_N \otimes \alpha_{P_L},$$

where α_{P_i} is a diagonal matrix with diagonal elements given by P_i . Next we are looking for states $|\psi\rangle$ for which the condition

$$\alpha_P|\psi\rangle = \lambda|\psi\rangle \quad (9)$$

is satisfied and, at the same time, $\mu([\psi]) = \alpha_P$. The problem of finding critical points is thus reduced to a study of eigenstates of the operator α_P which is an $N^L \times N^L$ diagonal matrix for $P \in \Psi(\mathbb{P}(\mathcal{H}))$. If the diagonal elements of α_P are nondegenerate the corresponding eigenvectors are separable states. Hence in order to find nontrivial SLOCC classes we are interested in the situation when the spectrum of α_P is degenerate. Remarkably, the dimensions of the degenerate eigenspaces are typically relatively small and hence it is quite easy to verify for which states $\mu([\psi]) = \alpha_P$. Finally, as we explain in [10] it is always possible to divide critical points found in this way into finite number of families, where each family contains SLOCC nonequivalent classes of states characterized by fixed reduced one-particle density matrices.

In order to demonstrate how to use the above method we provide calculations for two simple examples (three qubits and three 5-state fermions). For cases when calculation are more elaborate see [10].

Three qubits. For the three-qubit system the Hilbert space is $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, the SLOCC operations are represented by the group $G = SL(2)^{\times 3}$, and the local unitary operations by the group $K = SU(2)^{\times 3}$. Using the canonical form of a three-qubit state [11, 16],

$$|\psi\rangle = p|011\rangle + q|101\rangle + r|110\rangle + s|111\rangle + z|000\rangle, \quad (10)$$

the reduced one-qubit density matrices can be easily calculated. It is then a matter of straightforward calculations to see that there are exactly two states of the form (10) which have maximally mixed reduced one-particle density matrices. One of them is the GHZ state, $|\psi_{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, and the other one is LU equivalent to it. Effectively, the main SLOCC class is the G -orbit through GHZ . The remaining classes can be found by considering Equation (9), where α_P is 8×8 diagonal matrix. The Kirwan polytope is given by three polygonal inequalities [14] for the smallest eigenvalues of the reduced one-qubit density matrices and hence is three-dimensional. By direct calculations we find that the only eigenspaces contributing new critical points are

of dimension 1, 3 and 4. They give separable, biseparable and W SLOCC classes, respectively, i.e.

$$\begin{aligned} |\psi_W\rangle &= \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle), \\ |\psi_{BS1}\rangle &= \frac{1}{\sqrt{2}}(|100\rangle + |111\rangle), \quad \psi_{BS2} = \frac{1}{\sqrt{2}}(|010\rangle + |111\rangle), \\ |\psi_{BS3}\rangle &= \frac{1}{\sqrt{2}}(|001\rangle + |111\rangle), \quad \psi_{SEP} = |000\rangle. \end{aligned}$$

Since the state GHZ is in $\mu^{-1}(0)$ the Morse index at it is equal to zero [10]. This means it is impossible to increase the total variance Var by any perturbation of $K.[\psi_{GHZ}]$. The case of $[\psi_W]$ is more interesting, namely for the orthogonal complement of $\mathfrak{g}.[\psi_1]$ we have

$$(\mathfrak{g}.[\psi_1])^\perp = \text{Span}\{|111\rangle, i|111\rangle\}.$$

It is easy to check that the Hessian of the function $f(|\psi\rangle)$ defined by Eq. (8) is negative on $(\mathfrak{g}.[\psi_1])^\perp$ and hence the index equals 2. Similar calculations show that for biseparable states, i.e. $|\psi_{BS1}\rangle$, $|\psi_{BS2}\rangle$ and $|\psi_{BS3}\rangle$ we have $\text{index}(|\psi_{BSk}\rangle) = 6$ and for separable state $\text{index}(|\psi_{SEP}\rangle) = 8$.

Three five-state fermions. As a second example let us consider a system consisting of three fermions with five-dimensional single particle Hilbert spaces, $\mathcal{H} = \bigwedge^3(\mathbb{C}^5)$, $K = SU(5)$, $G = SL(5, \mathbb{C})$. In \mathbb{C}^5 we fix an orthonormal basis $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle, |5\rangle\}$ and chose the (also orthonormal) basis of $\bigwedge^3(\mathbb{C}^5)$

$$|i_1, i_2, i_3\rangle := |i_1\rangle \wedge |i_2\rangle \wedge |i_3\rangle, \quad 1 \leq i_1 < i_2 < i_3 \leq 5. \quad (11)$$

Any $|\psi\rangle \in \mathcal{H}$ has a decomposition

$$|\psi\rangle = \sum_{1 \leq i_1 < i_2 < i_3 \leq 5} c_{i_1, i_2, i_3} |i_1, i_2, i_3\rangle, \quad (12)$$

where the scalar coefficients fulfill the normalization condition. A system with $\mathcal{H} = \bigwedge^3(\mathbb{C}^5)$ can be considered as consisting of two holes and the structure of the spectrum of the rescaled density matrix $\rho([\psi])$ is well known [15]. First condition says that the eigenvalues of ρ are bounded from above by $\frac{1}{3}$. The second one is that the maximal eigenvalue λ_{max} equals exactly $\frac{1}{3}$. The last condition states that all eigenvalues except for λ_{max} are, at least, doubly degenerate. Using these properties it can be easily shown that there are only two nonequivalent critical sets of Var. They are parametrized by the states:

$$|\psi_1\rangle = |1, 2, 3\rangle, \quad |\psi_2\rangle = \frac{1}{\sqrt{2}}(|1, 2, 3\rangle + |1, 4, 5\rangle). \quad (13)$$

Using Theorem 2 we find, in the same manner as in the case of three qubits, that $\text{index}(|\psi_1\rangle) = 6$ and $\text{index}(|\psi_2\rangle) = 0$. The latter fact is a consequence of the observation that $(\mathfrak{g}.[\psi_2])^\perp = 0$. It follows that $G.[\psi_2]$ is

of the maximal dimension and that the total variance attains the global maximum on the K orbit passing through $[\psi_2]$. It is interesting to note that in the case considered $\mu^{-1}(0)$ is empty, that is there are no states in $\mathbb{P}(\mathcal{H})$ that have maximally mixed reduced density matrix.

Summarizing, we provided an algorithm for finding and classifying SLOCC equivalent pure states. The algorithm is effective and universally applicable in various setting including distinguishable and indistinguishable systems of particles with arbitrary number of components.

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After completion of this work, we have learned about independent related work by Walter, Doran, Gross, and Christandl [17].

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 - [18] We define Lie algebra \mathfrak{k} of a Lie group K by the physics convention: $X \in \mathfrak{k}$ if and only if $\exp(iX) \in K$. Therefore Lie algebra $\mathfrak{su}(N)$ consists of traceless hermitian opera-

tors.

[19] By the term “phase space structure” we actually mean

symplectic structure.